TAYLOR

Some Geometric Properties
Of Invariant Curves

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SOME GEOMETRIC PROPERTIES OF INVARIANT CURVES

BY

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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ENTITLED Some Geometric Properties of Invariant Curves.

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OF Bachelor of Arts.

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HEAD OF DEPARTMENT OF Mathematics.

Index.

Introduction.

Among the contributions of the nineteenth century to the mathematical knowledge of the world may be mentioned especially the advances made along the lines of analytic geometry and algebra. Of particular importance was the developement of the modern theory of curves and of invariants. These two subjects went hand in hand, one reacting on the other. It was Hesse (1811-74) who through his work gave a new impetus to the study of geometry, while to Cayley belongs the honor of expanding the theory of invariants from the purely algebraic side. Clebsch, who was a student of Hesse, finally combined the two and brought out most clearly the relation between geometry on the one side, and the theory of invariants on the other.

The subject of invariants was at first, about 1845, studied purely from the algebraic side. The fact, however, that a covariant involves also the variables soon led to a geometric interpretation of these functions, followed by the geometric interpretation of invariants proper. Thus a broad field for research was opened up and rapid strides were made, both on the algebraic and geometric side.

It is the purpose of this thesis to study from the geometric side certain properties of some of the most important covariant curves, particularly of the Hessian and Steinerian. The works referred to in this connection are chiefly-

Clebsch-Lindemann: Vorlesungen über Geometrie.

Durège: Die ebenen Curven dritter Ordnung.

Salmon: Higher Plane Curves.

At the end of the thesis is given a list, as complete as possible, of all journal articles bearing upon the subject.

§1. Definition of the Hessian, Steinerian and Cayleyan.

(a) Hessian.

The Hessian is defined as follows:-

1. Algebraically -

If we have given a single function of n variables $f(x_1, x_2, x_3, \text{----} x_n)$, the Hessian is the determinant formed of the second partial derivatives of the function with respect to the n variables, each member of the determinant being of the form $\frac{\partial^2 f}{\partial x_i \partial x_k}$ where $i = 1, 2, 3, \text{---} n$ and $k = 1, 2, 3 \text{---} n$. The Hessian determinant is then of the following form:-

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix}$$

2. Geometrically:—

The Hessian is the locus of all points whose polar conics relatively to the ground curve break up into pairs of right lines.

The equation of the Hessian may be obtained as follows. Since we are dealing with two dimensional space, we use a function of three variables $f(x_1, x_2, x_3)$. Then the polar conic of any point relatively to the curve is

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3} \right)^2 f(x_1, x_2, x_3,) = 0 \qquad (1)$$

wher the y's are the running coordinates and the x's are fixed. Equation (1) is of the second degree in the y's and hence is of the following form—

$$a y_1^2 + b y_2^2 + c y_3^2 + 2 f y_2 y_3 + 2 g y_3 y_1 + 2 h y_1 y_2 = 0 \qquad (2)$$

where

$$a = \frac{\partial^2 f}{\partial x_1^2}, \quad b = \frac{\partial^2 f}{\partial x_2^2}, \quad c = \frac{\partial^2 f}{\partial x_3^2}, \quad f = \frac{\partial^2 f}{\partial x_2 \partial x_3}, \quad g = \frac{\partial^2 f}{\partial x_3 \partial x_1}, \quad h = \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

The condition that the conic represented by equation (2) shall break up into a pair of right lines is that the following determi-

nant shall vanish.—

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \qquad (3)$$

or

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix} = 0 \qquad (4)$$

If now we let the x's become variables, equation (4) represents the locus of all points whose polar conics break up into pairs of right lines. This locus is the Hessian, and we see that it has the same form as when defined algebraically.

Example 1.

Let us find the Hessian of the following equation —

$$x^4 - 9x^2z^2 + y^3z = 0 \qquad (1)$$

We first compute the first partial derivatives which are -

$$\frac{\partial f}{\partial x} = -18xz^2 + 4x^3$$
$$\frac{\partial f}{\partial y} = 3y^2 \qquad (2)$$
$$\frac{\partial f}{\partial z} = y^3 - 18x^2$$

Then from these we form the second partial derivatives as follows -

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 18z^2 \qquad \frac{\partial^2 f}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial z \partial x} = -36xz$$
$$\frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 6yz \qquad \frac{\partial^2 f}{\partial z \partial y} = 3y^2 \qquad (3)$$
$$\frac{\partial^2 f}{\partial x \partial z} = 36xz \qquad \frac{\partial^2 f}{\partial y \partial z} = 3y^2 \qquad \frac{\partial^2 f}{\partial z^2} = -18x^2$$

Substituting these second partial derivatives in the determinant form, we have -

$$\begin{vmatrix} 12x^2 - 18z^2 & 0 & -36xz \\ 0 & 6yz & 3y^2 \\ -36xz & 3y^2 & -18x^2 \end{vmatrix} = 0 \qquad (4)$$

The expansion of this determinant gives us

$$y(-12x^4z + 90x^2z^3 - 2x^2y^3 + 3y^3z^2) = 0 \qquad (5)$$

the equation of the Hessian.

(b) Steinerian.

The Steinerian is the locus of all points whose first polars with respect to the ground curve have double points.

In order to find the equation of the Steinerian we proceed as follows. Form the first polar of a point (y_1, y_2, y_3) and express the condition that it shall have a double point. Eliminate the x's from these equations of condition and let the y's become variable We then have the equation of the locus of all points whose first polars have double points, or the equation of the Steinerian.

Example 2.

Let us find the Steinerian of the following equation –

$$x^3 + y^3 - 3xyz = 0 \qquad (1)$$

Let (α, β, γ) be any point on the Steinerian. Its first polar, expressed symbolically, is –

$$\left(\alpha\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial y}+\gamma\frac{\partial}{\partial z}\right)f(x,y,z)=0 \qquad (2)$$

Forming the partial derivatives and substituting them in equation (2) it becomes

$$\alpha(x^2-yz)+\beta(y^2-xz)-\gamma xy=0=\phi(xyz) \qquad (3)$$

The condition that this first polar has a double point is as follows:

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=2\alpha x-\beta z-\gamma y=0\\ \frac{\partial\phi}{\partial y}&=-\alpha z+2\beta y-\gamma x=0\\ \frac{\partial\phi}{\partial z}&=-\alpha y-\beta x=0\end{aligned} \qquad (4)$$

Eliminating x, y and z from these equations (4) we have

$$\begin{vmatrix}2\alpha & -\gamma & -\beta\\ -\gamma & 2\beta & -\alpha\\ -\beta & -\alpha & 0\end{vmatrix}=0 \qquad (5)$$

Expanding this determinant and letting α, β and γ become variables we have the equation of the steinerian

$$x^3+y^3+xyz=0 \qquad (6)$$

Example 3.

We will now find the Steinerian of Example 1., page 5.

$$x^4 - 9x^2z^2 + y^3z = 0 \qquad (1)$$

The first polar of any point (ξ, η, ζ) is

$$\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}\right) f(x, y, z) = 0 \qquad (2)$$

Substituting in this equation (2 the partial derivatives found in Example 1. we have

$$\xi(4x^3 - 18xz^2) + \eta(3y^2z) + \zeta(y^3 - 18x^2z) = 0 = \phi(xyz) \qquad (3)$$

The condition that this first polar has a double point is the following

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= 12\xi x^2 - 18\xi z^2 - 36\zeta x z &= 0 \\ \frac{\partial \phi}{\partial y} &= 6\eta y z + 3\zeta y^2 &= 0 \\ \frac{\partial \phi}{\partial z} &= -36\xi x z + 3\eta y^2 - 18\zeta x^2 &= 0 \end{aligned} \qquad (4)$$

Dividing thro' by z^2, these equations (4) become

$$2\xi \frac{x^2}{z^2} - 3\xi - 6\zeta \frac{x}{z} = 0 \qquad (a)$$

$$2\eta \frac{y}{z} + \zeta \frac{y^2}{z^2} = 0 \qquad (b)$$

$$-12\xi \frac{x}{z} + \eta \frac{y^2}{z^2} - 6\zeta \frac{x^2}{z^2} = 0 \qquad (c)$$

From (b), $\frac{y}{z} = -\frac{2\eta}{\zeta}$. Making this substitution in

(c) it becomes

$$-12\,\xi\,\frac{x}{z} + 4\,\frac{\eta^3}{\xi^2} - 6\,\xi\,\frac{x^2}{z^2} = 0 \qquad (d)$$

We wish to eliminate x, y and z but we have only two equations (a) and (d) in four variables. So we multiply each by $\frac{x}{z}$ thus obtaining four equations in three variables. Their eliminant is

$$\begin{vmatrix} 2\xi & -6\xi & -3\xi & 0 \\ -6\xi & -12\xi & 4\frac{\eta^3}{\xi^2} & 0 \\ 0 & 2\xi & -6\xi & -3\xi \\ 0 & -6\xi & -12\xi & +\frac{\eta^3}{\xi^2} \end{vmatrix} = 0 \qquad (5)$$

Expanding this determinant and substituting for ξ, η and ξ the running coordinates (x, y, z) we have the equation of the Steinerian

$$243\,z^4x^2 + 216\,z^5y^3 + 216\,z^7x^4 + 216\,z^3x^2y^3 - 16x^2y^6 = 0$$

We have seen that the Steinerian is the locus of points whose first polars have double points, and will see later on that the Hessian is the locus of these double points. So for every point

A on the Steinerian, there is a point A', on the Hessian, which is the double point of the first polar of A. Such points are called corresponding points.

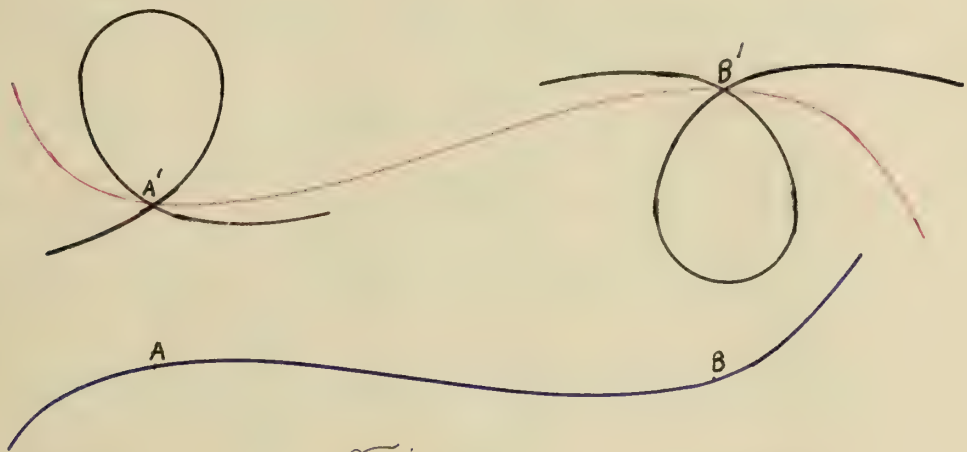

Fig. 1.

(c) Cayleyan.

If we connect the point A on the Steinerian with its corresponding point A' on the Hessian, and consider the point A as moving along the Steinerian, we then have a system of lines joining corresponding points. This system of lines has an envelope which is called the Cayleyan of the ground curve.

The method of finding the equation of the Cayleyan may be outlined as follows. Take any point (α, β, γ) on the Steinerian. Find the coordinates of the double point of its first polar by solving the equations of condition such as (4) in examples 2. and 3. Write the equation of the line thro' (α, β, γ) and the corresponding point thus obtained. Let (α, β, γ) become variable. We then have represented a system of lines joining corresponding points. Finding the envelope of this system of lines, we have the Cayleyan of the ground curve.

§ 2. Properties of the Hessian and Steinerian.

Theorem 1. – The Hessian is a covariant.

Proof: –

In order to prove that the Hessian is a covariant, we subject the original function, $f(x_1, x_2 \cdots x_n) = 0$ to linear transformation and show that the Hessian of the transformed function is equal to the Hessian of the original function multiplied by some power of the modulus of transformation. We have given –

$$f(x_1, x_2, x_3, \cdots x_n) = 0 \qquad (1)$$

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \frac{\partial^2 f}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{vmatrix} = H \qquad (2)$$

which we now subject to the following linear

transformation -

$$
\begin{aligned}
x_1 &= \alpha_{11}\bar{x}_1 + \alpha_{12}\bar{x}_2 + \alpha_{13}x_3 + \cdots + \alpha_{1n}\bar{x}_n \\
x_2 &= \alpha_{21}\bar{x}_1 + \alpha_{22}\bar{x}_2 + \alpha_{23}x_3 + \cdots + \alpha_{2n}\bar{x}_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_n &= \alpha_{n1}\bar{x}_1 + \alpha_{n2}\bar{x}_2 + \alpha_{n3}\bar{x}_3 \cdots\cdots + \alpha_{nn}\bar{x}_n
\end{aligned}
\qquad (3)
$$

where the modulus of transformation is

$$
M = \begin{vmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn}
\end{vmatrix} = 0 \qquad (4)
$$

If we denote the transformed function by

$$F(\bar{x}_1, \bar{x}_2, \bar{x}_3 \cdots \bar{x}_n) = 0$$

then the Hessian of the transformed function is

$$
\begin{vmatrix}
\frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\
\frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} & \frac{\partial^2 F}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 F}{\partial x_2 \partial x_n} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial^2 F}{\partial x_n \partial x_1} & \frac{\partial^2 F}{\partial x_n \partial x_2} & \frac{\partial^2 F}{\partial x_n \partial x_3} & \cdots & \frac{\partial^2 F}{\partial x_n^2}
\end{vmatrix} = H' \qquad (5)
$$

In order to evaluate H', we must find $\frac{\partial^2 F}{\partial x_i \partial x_K}$ where $i = 1, 2, 3, \ldots n$ and $K = 1, 2, 3, \ldots n$. To do this we build it up as follows -

$$\frac{\partial F}{\partial \bar{x}_i} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial \bar{x}_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial \bar{x}_i} + \cdots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i}$$

but

$$\frac{\partial x_i}{\partial \bar{x}_K} = \alpha_{iK}$$

then

$$\frac{\partial F}{\partial \bar{x}_i} = \frac{\partial F}{\partial x_1} \alpha_{1i} + \frac{\partial F}{\partial x_2} \alpha_{2i} + \cdots + \frac{\partial F}{\partial x_n} \alpha_{ni} = P_i$$

and

$$\frac{\partial^2 F}{\partial \bar{x}_i^2} = \frac{\partial P_i}{\partial \bar{x}_i} = \frac{\partial P_i}{\partial x_1} \alpha_{1i} + \frac{\partial P_i}{\partial x_2} \alpha_{2i} + \cdots + \frac{\partial P_i}{\partial x_n} \alpha_{nK}$$

Therefore

$$\frac{\partial^2 F}{\partial x_i \partial x_K} = \frac{\partial P_i}{\partial \bar{x}_K} = \frac{\partial P_i}{\partial x_1} \alpha_{1K} + \frac{\partial P_i}{\partial x_2} \alpha_{2K} + \cdots + \frac{\partial P_i}{\partial x_n} \alpha_{nK}$$

Forming the Hessian from these new derivatives we have

$$\begin{vmatrix} \left(\frac{\partial P_1}{\partial x_1}\alpha_{11} + \frac{\partial P_1}{\partial x_2}\alpha_{21} + \cdots + \frac{\partial P_1}{\partial x_n}\alpha_{n1}\right), & \cdots & \left(\frac{\partial P_1}{\partial x_1}\alpha_{1n} + \frac{\partial P_1}{\partial x_2}\alpha_{2n} + \cdots + \frac{\partial P_1}{\partial x_n}\alpha_{nn}\right) \\ \left(\frac{\partial P_2}{\partial x_1}\alpha_{11} + \frac{\partial P_2}{\partial x_2}\alpha_{21} + \cdots + \frac{\partial P_2}{\partial x_n}\alpha_{n1}\right), & \cdots & \left(\frac{\partial P_2}{\partial x_1}\alpha_{1n} + \frac{\partial P_2}{\partial x_2}\alpha_{2n} + \cdots + \frac{\partial P_2}{\partial x_n}\alpha_{nn}\right) \\ \cdots & \cdots & \cdots \\ \left(\frac{\partial P_n}{\partial x_1}\alpha_{11} + \frac{\partial P_n}{\partial x_2}\alpha_{21} + \cdots + \frac{\partial P_n}{\partial x_n}\alpha_{n1}\right), & \cdots & \left(\frac{\partial P_n}{\partial x_1}\alpha_{1n} + \frac{\partial P_n}{\partial x_2}\alpha_{2n} + \cdots + \frac{\partial P_n}{\partial x_n}\alpha_{nn}\right) \end{vmatrix} = H' \qquad (6)$$

which is made up of the following factors

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial P_1}{\partial X_1} & \frac{\partial P_1}{\partial X_2} & \cdots & \frac{\partial P_1}{\partial X_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_n}{\partial X_1} & \frac{\partial P_n}{\partial X_2} & \cdots & \frac{\partial P_n}{\partial X_n} \end{vmatrix}$$

or

$$M \cdot \begin{vmatrix} \frac{\partial P_1}{\partial X_1} & \frac{\partial P_1}{\partial X_2} & \cdots & \frac{\partial P_1}{\partial X_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial P_n}{\partial X_1} & \frac{\partial P_n}{\partial X_2} & \cdots & \frac{\partial P_n}{\partial X_n} \end{vmatrix} \qquad (7)$$

now

$$\frac{\partial P_1}{\partial X_1} = \frac{\partial^2 F}{\partial X_1^2}\alpha_{11} + \frac{\partial^2 F}{\partial X_2 \partial X_1}\alpha_{21} + \frac{\partial^2 F}{\partial X_3 \partial X_1}\alpha_{31} + \cdots + \frac{\partial^2 F}{\partial X_n \partial X_1}\alpha_{n1}$$

$$\frac{\partial P_2}{\partial X_1} = \frac{\partial^2 F}{\partial X_1^2}\alpha_{12} + \frac{\partial^2 F}{\partial X_2 \partial X_1}\alpha_{22} + \frac{\partial^2 F}{\partial X_3 \partial X_1}\alpha_{32} + \cdots + \frac{\partial^2 F}{\partial X_n \partial X_1}\alpha_{n1}$$

$\vdots$

etc.

Hence by substituting these values for the derivatives in equation (7) we have

$$M \cdot \begin{vmatrix} \left(\frac{\partial^2 F}{\partial X_1^2}\alpha_{11} + \frac{\partial^2 F}{\partial X_2 \partial X_1} + \cdots + \frac{\partial^2 F}{\partial X_n \partial X_1}\alpha_{n1}\right), & \cdots & \left(\frac{\partial^2 F}{\partial X_n \partial X_1}\alpha_{11} + \frac{\partial^2 F}{\partial X_n \partial X_2}\alpha_{21} + \cdots + \frac{\partial^2 F}{\partial X_n^2}\alpha_{n1}\right) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \left(\frac{\partial^2 F}{\partial X_1^2}\alpha_{1n} + \frac{\partial^2 F}{\partial X_2 \partial X_1}\alpha_{2n} + \cdots + \frac{\partial^2 F}{\partial X_n \partial X_1}\alpha_{nn}\right), & \cdots & \left(\frac{\partial^2 F}{\partial X_n \partial X_1}\alpha_{1n} + \frac{\partial^2 F}{\partial X_n \partial X_2}\alpha_{2n} + \cdots + \frac{\partial^2 F}{\partial X_n^2}\alpha_{nn}\right) \end{vmatrix} = H' \qquad (8)$$

This determinant in turn is the product of two determinants, one of which again is the Modulus. Therefore we have

$$M^2 \cdot \begin{vmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_1} \\ \frac{\partial^2 F}{\partial x_1 \partial x_2} & \frac{\partial^2 F}{\partial x_2^2} & \cdots & \frac{\partial^2 F}{\partial x_n \partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 F}{\partial x_1 \partial x_n} & \frac{\partial^2 F}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 F}{\partial x_n^2} \end{vmatrix} = H' \qquad (9)$$

Since the form of the F function is the same as that of the f function we can write f for F in equation (9). Hence we have

$$H' = M^2 H.$$

This covariant property of the Hessian may be interpreted geometrically as follows. Suppose we have a curve $F=0$ and its Hessian $H=0$. If we subject the curve $F=0$ to linear transformation it passes over into a new curve $F'=0$. The Hessian also passes

over into a new curve $H'=0$ which bears the same relation to $F'=0$ that $H=0$ does to $F=0$. Thus, as will be shown later, the Hessian $H=0$ passes thro' all the double points and points of inflection of the ground curve $F=0$ and the Hessian $H'=0$ also passes thro' all the double points and points of inflection of the transformed curve $F'=0$.

The Order of the Hessian.

If the ground curve is of degree n, each of the second partial derivatives which enter into the equation of the Hessian is of degree $n-2$. Since the Hessian is the expansion of a three row determinant, it is therefore of degree $3(n-2)$. Hence we see that no curve of degree lower than three can have a Hessian.

The Intersections of the Hessian with the Ground Curve.

We know that two curves of degree m and n respectively intersect in mn points. Hence, since the ground curve is of degree n and the Hessian of degree $3(n-2)$, the two intersect in $3n(n-2)$ points.

Theorem 2. — The Hessian cuts the ground curve only in its double points and points of inflection.

Proof:-

It is known that the polar conic of any point of inflection or of any double point of the ground curve breaks up into a pair of right lines. This does not happen in the case of the polar conic of an ordinary point, which may be easily shown as follows. At an ordinary point, the polar line is tangent to the curve, and all higher polars are tangent to the polar line.

Now, if the polar conic were to break up into a pair of right lines, the tangent at the point would have to be one of these, that is, the first polar would be a factor of the second polar. This is, however, the condition for a point of inflection. It follows then that the polar conics of all points of inflection and double points of the curve break up into pairs of right lines, while the polar conics of all ordinary points do not. By definition, the polar conic of every point of the Hessian breaks up into a pair of right lines. Hence it follows that the Hessian can cut the ground curve only in its points of inflection and double points.

Example 4.

Let us consider the curve

$$x^4 - 9x^2z^2 + y^3z = 0 \qquad (1)$$

which is the curve of Example 1. page 5.

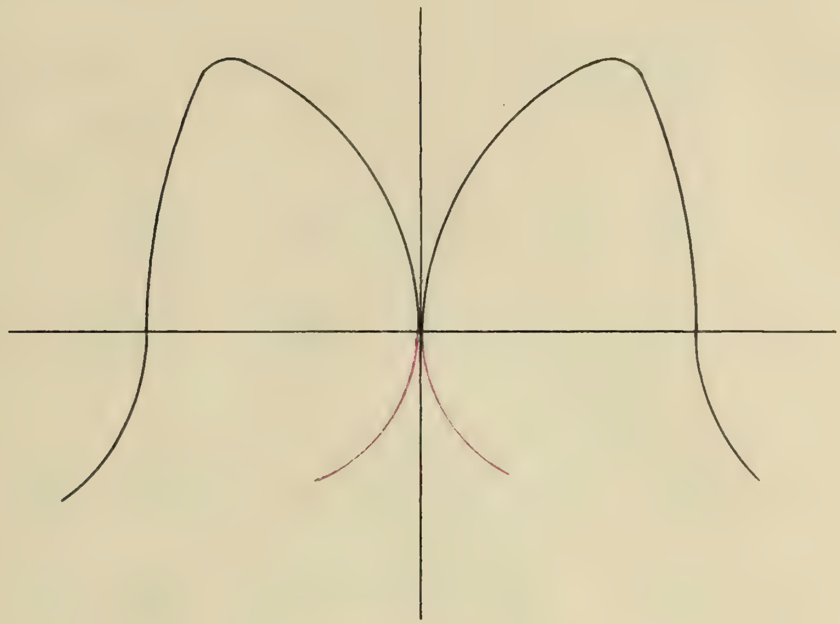

Fig. 2.

The highest power of z entering into equation (1) is the second or $(n-2)$nd. Hence the curve has a double point at the origin. The tangents at this double point are given by the coefficient of z^2, that is, by $x^2 = 0$. Hence the curve has a cusp at the origin, the cuspidal tangent being the y axis. To further study this curve we let $z = 1$. Then equation (1) becomes

$$y^3 = x^2(9 - x^2) \qquad (2)$$

The intercepts of this curve on the y axis are (0,0) and on the x axis (3,0) and (−3,0). To test for maximum or minimum points $\frac{dy}{dx} = 0$ is solved. Thus

$$\frac{dy}{dx} = \frac{18 - 12x^2}{3y^2} = 0 \qquad (3)$$

$$18 - 12x^2 = 0$$

$$\therefore \quad x = \pm 3\sqrt{\tfrac{1}{2}}$$

Substituting this value of x in the original equation (2), we find that

$$y = 2\tfrac{4}{5} \text{ approximately.}$$

Hence the curve has two maximum points,— $(3\sqrt{\tfrac{1}{2}}, 2\tfrac{4}{5})$ and $(-3\sqrt{\tfrac{1}{2}}, 2\tfrac{4}{5})$ which shows that the cusp extends above the x axis. To test for points of inflection $\frac{d^2y}{dx^2} = 0$ is solved as follows

$$y = x^{\frac{2}{3}}(9-x^2)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}(9-x^2)^{\frac{1}{3}} - \frac{2}{3}x^{\frac{5}{3}}(9-x^2)^{-\frac{2}{3}} \qquad (4)$$

$$\frac{d^2y}{dx^2} = -\frac{2}{9}x^{-\frac{4}{3}}(9-x^2)^{\frac{1}{3}} - \frac{10}{9}x^{\frac{2}{3}}(9-x^2)^{-\frac{2}{3}} - \frac{4}{9}x^{\frac{2}{3}}(9-x^2)^{-\frac{2}{3}} - \frac{8}{9}x^{\frac{8}{3}}(9-x^2)^{-\frac{5}{3}} = 0$$

$$= -\frac{2}{9}x^{-\frac{4}{3}}(9-x^2) - \frac{14}{9}x^{\frac{2}{3}} - \frac{8}{9}x^{\frac{8}{3}}(9-x^2)^{-1} = 0$$

multiplying thro' by $(9-x^2)^2$ we have

$$\frac{d^2y}{dx^2} = -\frac{2}{9}x^{-\frac{4}{3}}(9-x^2)^3 - \frac{4}{9}x^{\frac{2}{3}}(9-x^2)^2 - \frac{8}{9}x^{\frac{8}{3}}(9-x^2) = 0 \qquad (5)$$

$\therefore\ x = \pm 3$, and the intercepts on the x axis are points of inflection. The curve has no asymptotes, for as x increases beyond $\pm 3\sqrt{\frac{1}{2}}$ y increases indefinitely.

The Hessian of this curve has in Example 1. been shown to be

$$y(-12x^4z + 90x^2z^3 - 2x^2y^3 + 3y^3z^2) = 0 \qquad (6)$$

We see from the form of this equation that the Hessian curve consists of two branches, one, the x axis and the other with a cusp at the origin having the same tangent as the cusp of the ground curve. In order to find in which direction the cusp extends, we substitute positive values of x in the equation of the Hessian and find that for all values of x between 0 and approximately 3, y is negative. Therefore the cusp extends downward.

Theorem 3. - The Hessian has a node at every nodal point of the ground curve, both curves having the same tangents.

Proof: -

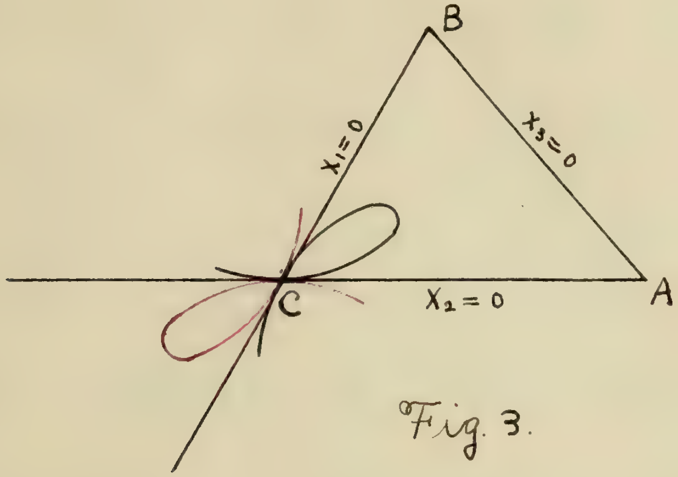

Fig. 3.

If we arrange the homogeneous equation of the mth degree in x_1, x_2 and x_3 according to descending powers of x_3, then the sufficient condition that the curve has a double point is that the highest power of x_3 present shall be x_3^{m-2}. In this case the equation of the curve is

$$f^{(2)}x_3^{m-2}+f^{(3)}x_3^{m-3}+----+f^{(i)}x_3^{m-i}+----+f^{(m)}=0 \qquad (1)$$

where $f^{(i)}$ is a homogeneous function of degree i in x_1 and x_2 together. In order to

simplify the equation of the curve, we take the vertex C of the triangle of reference as the node and the sides $x_1 = 0$ and $x_2 = 0$ as the tangents at the node. Then, since the coefficient of x_3^{m-2} represents the tangents at the node, the equation of the curve reduces to

$$x_1 x_2 x_3^{m-2} + f^{(3)} x_3^{m-3} + \text{-----} = 0 \tag{2}$$

We then compute the partial derivatives of this equation in order to obtain the Hessian. Its equation is the following

$$\begin{vmatrix} \dfrac{\partial^2 f^{(3)}}{\partial x_1^2} x_3^{m-3}, & x_3^{m-2} + \dfrac{\partial^2 f^{(3)}}{\partial x_1 \partial x_2} x_3^{m-3}, & (m-2) x_2 x_3^{m-3} + (m-3) \dfrac{\partial f^{(3)}}{\partial x_1} x_3^{m-4} \\ x_3^{m-2} + \dfrac{\partial^2 f^{(3)}}{\partial x_1 \partial x_2} x_3^{m-3}, & \dfrac{\partial^2 f^{(3)}}{\partial x_2^2} x_3^{m-3}, & (m-2) x_1 x_3^{m-3} + (m-3) \dfrac{\partial f^{(3)}}{\partial x_2} x_3^{m-4} \\ (m-2) x_2 x_3^{m-3} + (m-3) \dfrac{\partial f^{(3)}}{\partial x_2} x_3^{m-4}, & (m-2) x_1 x_3^{m-3} + (m-3) \dfrac{\partial f^{(3)}}{\partial x_1} x_3^{m-4}, & (m-2)(m-3) x_1 x_2 x_3^{m-4} \end{vmatrix} = 0$$

where $f^{(3)} = \alpha x_1^3 + 3\beta x_1^2 x_2 + 3\gamma x_1 x_2^2 + \delta x_2^3$ (3)

Expanding this determinant and arranging according to descending powers of x we have

$$H = (m-2)(m-1) x_1 x_2 x_3^{3m-8} + (m-1)\left[2(m-2) x_1 x_2 \frac{\partial^2 f^{(3)}}{\partial x_1 \partial x_2} - m f^{(3)}\right] x_3^{3m-9}$$

$+ \text{-------} = 0 \qquad (4)$

In this equation we must have $m > 2$, otherwise we should have no Hessian. Hence it follows that the first term of this equation (4) cannot vanish identically. Now the degree of the Hessian is $3m-6$, but the highest power of x_3 in equation (3) is x_3^{3m-8}. Hence we see that the Hessian must have a double point at the vertex C. Moreover, since the tangents to the Hessian, given by the coefficient of x_3^{3m-8} are $x_1 = 0$ and $x_2 = 0$, the Hessian has the same tangents as the ground curve at the double point.

<u>Theorem 4.</u> — At a node, a branch of the Hessian and the tangent branch of the ground curve are convex toward each other.

Proof: —

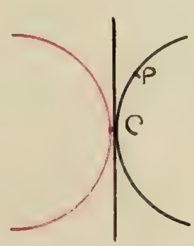

Fig. 4.

We take the branches of the two curves tangent to the side $x_1=0$ of the fundamental triangle. Let P be any point on the ground curve. Its position is determined by the equation

$$x_1 = \lambda x_2 \qquad (1)$$

Now as the point P moves indefinitely close to the vertex C, λ becomes infinitesimal. It follows that x_1 is of higher order of infinitesimal than x_2. Then $f^{(3)}$ (see equation (1) p 25 and (3) p 26) reduces to

$$f^{(3)} = \delta x_2^{(3)} \qquad (2)$$

Then the equation of the curve in the neighborhood of the point C reduces to

$$x_1 x_2 x_3^{m-2} + \delta x_2^3 x_3^{m-3} = 0 \qquad (3)$$

or

$$x_2 (x_1 x_3 + \delta x_2^2) = 0 \qquad (4)$$

If we consider the Hessian in the same way, its equation in the neighborhood of the point C reduces to

$$x_1 x_3 - m \delta x_2^2 = 0 \qquad (5)$$

Equations (4) and (5) are both of parabolic type and correspond respectively to the parabolas $y^2 = 4px$ and $y^2 = -4px$ whose convexities are in opposite directions. Therefore the two curves or tangent branches represented by these two equations lie on opposite sides of the tangent line.

Example 5.

Let us consider the curve

$$x^3 + y^3 = 3xyz \qquad (1)$$

which is the curve of Example 2, page 7.

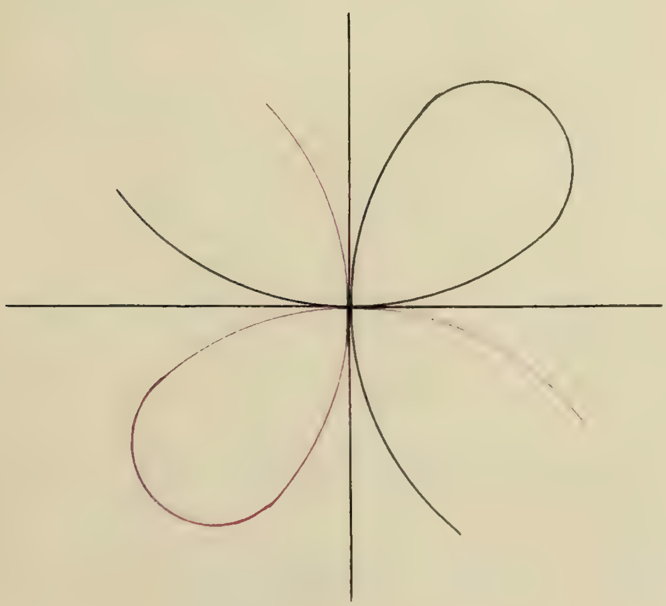

Fig. 5.

The highest power of z entering into equation (1) is the first or $(n-2)$nd. Hence the curve has a **nodal** point at the origin. The tangents at the double point, given by the coefficient of z, are the lines $x=0$ and $y=0$.

Let us find the equation of the Hessian of this curve. Going thro' the usual process of finding derivatives, we have.

$$H = \begin{vmatrix} 6x & -3z & -3y \\ -3z & 6y & -3x \\ -3y & -3x & 0 \end{vmatrix} = 0 \qquad (2)$$

The expansion of this determinant is

$$x^3 + y^3 + xyz = 0 \qquad (3)$$

The form of this equation is the same as that of the ground curve. Therefore the Hessian also has a node at the origin. From Theorem 4. we know that the Hessian and the ground curve have the relative

positions of Fig. 5. that is, a branch of the one is convex to the tangent branch of the other.

This example serves to illustrate the facts (1) that the Hessian passes thro' the double points of the ground curve, (2) that at every nodal point of the ground curve, the Hessian has a nodal point, both curves having the same tangent, (3) that at the double point a branch of the ground curve is convex to a tangent branch of the Hessian.

Theorem 5.- The Hessian has a triple point at the cusp of the ground curve, and two of its branches are tangent to the tangent at the cusp, while the third branch is independent.

Proof:-

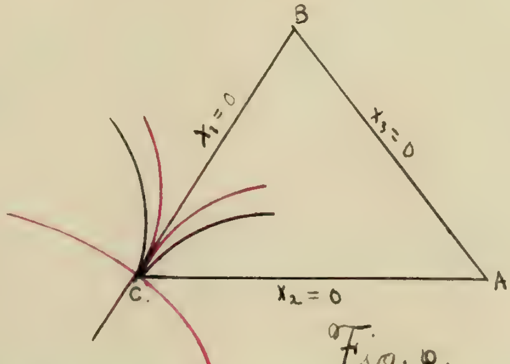

Fig. 10.

If a curve of the mth degree has a cusp, and the vertex C of the triangle of reference is taken at the cusp, while at the same time the line $x_1 = 0$ is the cuspidal tangent, then the equation of the curve reduces to

$$x_1^2 x_3^{m-2} + \phi^{(3)} x_3^{m-3} + \phi^{(4)} x_3^{m-4} + \cdots = 0 \qquad (1)$$

By a suitable choice of the side $x_2 = 0$, equation (1) may be reduced to the form

$$x_1^2 x_3^{m-2} + \delta x_2^3 x_3^{m-3} + \cdots = 0 \qquad (2)$$

Equation (2) again may be further simplified by letting $x_2' = \delta^{\frac{1}{3}} x_2$. It then becomes

$$x_1^2 x_3^{m-2} + x_2^3 x_3^{m-3} + f^{(4)} x_3^{m-4} + \cdots = 0 \qquad (3)$$

We will now compute the values of the second partial derivatives of this equation (3) in order to find its Hessian. We have then

$$a = 2x_3^{m-2} + \frac{\partial^2 f^{(4)}}{\partial x_1 \partial x_2} x_3^{m-4} + \cdots$$

$$b = 6x_2 x_3^{m-3} + \frac{\partial^2 f^{(4)}}{\partial x_2^2} x_3^{m-4} + \cdots$$

$$c = (m-2)(m-3) x_1^2 x_3^{m-4} + (m-3)(m-4) x_2^3 x_3^{m-5} + \cdots$$

$$f = 3(m-3)x_2^2 x_3^{m-4} + (m-4)\frac{\partial f^{(4)}}{\partial x_1} x_3^{m-5} + \text{------}$$

$$g = 2(m-2)x_1 x_3^{m-3} + (m-4)\frac{\partial f^{(4)}}{\partial x_1} x_3^{m-5} + \text{------}$$

$$h = \frac{\partial f^{(4)}}{\partial x_1 \partial x_2} x_3^{m-4} + \frac{\partial f^{(5)}}{\partial x_1 \partial x_2} x_3^{m-5} + \text{--------}$$

Substituting these values in the determinant form (See equation (3), p. 5.), expanding and arranging according to descending powers of x_3, the equation of the Hessian becomes

$$-12(m-2)(m-1)x_1^2 x_2 x_3^{3m-9} - (m-1)\left[6(m-3)x_2^4 + x_1^2\frac{\partial f^{(4)}}{\partial x_2^2}\right]x_3^{3m-10}$$

$$+ \text{------} = 0 \qquad (4)$$

The first term of equation (4) cannot vanish identically when $m > 2$. Then, since the highest power of x_3 is the $(3m-9)$th, that is, is three less than the degree of the equation, the condition for a triple point is fulfilled. The Hessian has then a triple point at the cusp of the ground curve. The tangents to the Hessian at the point

C are given by the coefficient of x_3^{3m-9}. that is by $x_1^2 = 0$ and $x_2 = 0$. The coincident tangents $x_1^2 = 0$ are the same as the cuspidal tangent of the ground curve, but the tangent $x_2 = 0$ is independent. Therefore two branches of the Hessian are tangent to the ground curve at C, while the third branch is independent and cuts the ground curve in two points at C.

This theorem has already been illustrated in Example 4. page 21.

Example 6.

Let us consider the following curve

$$x^3 + xy^2 - y^2 z = 0 \qquad (1)$$

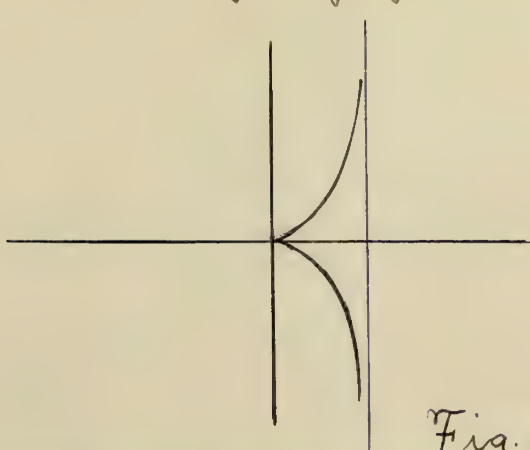

Fig. 7.

Since the highest power of z entering into equation (1) is the $(n-2)$nd, we see that the curve has a double point at the origin. The tangents at the double point are given by $y^2=0$. Hence the double point is a cusp, whose cuspidal tangent is the x axis. If we let $z=1$ and give to x negative values, y is imaginary. Hence the curve lies wholly on the positive side of the y axis. For any value of x higher than 1, y is imaginary.

Let us now find the Hessian of this curve. Obtaining the second partial derivatives as usual we have

$$H = \begin{vmatrix} 6x & 2y & 0 \\ 2y & 2x-2z & -2y \\ 0 & -2y & 0 \end{vmatrix} = 0 \qquad (2)$$

The expansion of this determinant is

$$H = xy^2 = 0 \qquad (3)$$

Hence we see that the Hessian is com-

posed of the x axis taken twice and the y axis taken once. This example illustrates the fact that at a cusp, the Hessian has a triple point, and two of its branches ($y^2=0$) are tangent to the ground curve while the third ($x=0$) is independent.

Theorem 6.- In general, the Hessian has no double points if the coefficients of the ground curve are independent of each other.

Proof:-

The condition that any curve shall have a double point is that its first partial derivatives shall vanish. In this case, these partial derivatives cannot be independent of each other, so that some relation must exist between the coefficients of the equation of the curve. The condition that the Hessian shall have a double point is, again, that its first partial deriva-

tives shall vanish. This also requires that some relation exists between the coefficients of the Hessian. But these coefficients are obtained from and depend upon the coefficients of the ground curve. Hence, in order that the Hessian have a double point, some relation must exist between the coefficients of the ground curve.

Theorem 7.— If a straight line forms a part of a curve $u=0$ of the nth order, it also forms a part of the Hessian. (Durège: Die Ebenen Curven dritter Ordnung).

Proof:—

If we take the straight line as the side $x_1=0$ of the fundamental triangle, the equation of the curve may be written thus

$$u = x_1 v \tag{1}$$

where v is a function of order $(n-1)$.

We will now find the Hessian of this equation. The first partial derivatives are

$$\frac{\partial u}{\partial x_1} = v + x_1 \frac{\partial v}{\partial x_1}$$

$$\frac{\partial u}{\partial x_2} = x_1 \frac{\partial v}{\partial x_2}$$

$$\frac{\partial u}{\partial x_3} = x_1 \frac{\partial v}{\partial x_3}$$

Forming the second partial derivatives, and substituting in the determinant form we have

$$H = \begin{vmatrix} 2\frac{\partial v}{\partial x_1} + x_1 \frac{\partial^2 v}{\partial x_1^2} & \frac{\partial v}{\partial x_2} + x_1 \frac{\partial^2 v}{\partial x_1 \partial x_2} & \frac{\partial v}{\partial x_3} + x_1 \frac{\partial^2 v}{\partial x_1 \partial x_3} \\ \frac{\partial v}{\partial x_2} + x_1 \frac{\partial^2 v}{\partial x_1 \partial x_2} & \frac{\partial^2 v}{\partial x_2^2} & \frac{\partial^2 v}{\partial x_2 \partial x_3} \\ \frac{\partial v}{\partial x_3} + x_1 \frac{\partial^2 v}{\partial x_1 \partial x_3} & \frac{\partial^2 v}{\partial x_3 \partial x_2} & \frac{\partial^2 v}{\partial x_3^2} \end{vmatrix}$$

If we separate this determinant into two sums according to the elements of the last column, we have

$$H = \begin{vmatrix} 2\dfrac{\partial v}{\partial x_1} + x_1\dfrac{\partial^2 v}{\partial x_1^2} & \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & \dfrac{\partial v}{\partial x_3} \\[2ex] \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & x_1\dfrac{\partial^2 v}{\partial x_2^2} & 0 \\[2ex] \dfrac{\partial v}{\partial x_3} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_3} & x_1\dfrac{\partial^2 v}{\partial x_3 \partial x_2} & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 2\dfrac{\partial v}{\partial x_1} + x_1\dfrac{\partial^2 v}{\partial x_1^2} & \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_3} \\[2ex] \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & x_1\dfrac{\partial^2 v}{\partial x_2^2} & x_1\dfrac{\partial^2 v}{\partial x_2 \partial x_3} \\[2ex] \dfrac{\partial v}{\partial x_3} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_3} & x_1\dfrac{\partial^2 v}{\partial x_3 \partial x_1} & x_1\dfrac{\partial^2 v}{\partial x_3^2} \end{vmatrix}$$

But both terms of this sum contain the factor x_1. Hence we have

$$H = x_1 \left\{ \frac{\partial v}{\partial x_3} \begin{vmatrix} \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & \dfrac{\partial^2 v}{\partial x_2^2} \\[2ex] \dfrac{\partial v}{\partial x_3} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_3} & \dfrac{\partial^2 v}{\partial x_3 \partial x_2} \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} 2\dfrac{\partial v}{\partial x_1} + x_1\dfrac{\partial^2 v}{\partial x_1^2} & \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & \dfrac{\partial^2 v}{\partial x_1 \partial x_3} \\[2ex] \dfrac{\partial v}{\partial x_2} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_2} & x_1\dfrac{\partial^2 v}{\partial x_2^2} & \dfrac{\partial^2 v}{\partial x_2 \partial x_3} \\[2ex] \dfrac{\partial v}{\partial x_3} + x_1\dfrac{\partial^2 v}{\partial x_1 \partial x_3} & x_1\dfrac{\partial^2 v}{\partial x_3 \partial x_2} & \dfrac{\partial^2 v}{\partial x_3^2} \end{vmatrix} \right\}$$

Therefore the straight line $x_1 = 0$ forms a part of the Hessian.

Example 7.

Let us consider the curve given by

$$x^4 + xy^3 - 3x^2yz = 0 \qquad (1)$$

which consists of the curve

$$x^3 + y^3 - 3xyz = 0$$

and the straight line

$$x = 0$$

Obtaining the Hessian of this curve in the usual manner we have

$$\begin{vmatrix} 2x^2 - yz & y^2 - 2xz & -2xz \\ xy - xz & 2xy & -x^2 \\ xy & 1^2 & 0 \end{vmatrix} = 0$$

or

$$2x^6 + 2x^4z^2 - x^4yz + 4x^3y^2z - x^3y^3 = 0$$

This we see consists of the curve

$$2x^3 + 2xz^2 - xyz + 4y^2z - y^3 = 0$$

and the line $\quad x = 0$.

<u>Theorem 8.</u> – The Hessian is the locus of the double points of first polars. (Fig 1. p11.)

Proof:–

Let (y_1, y_2, y_3) be any fixed point. Its first polar is

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3}\right) f(x_1, x_2, x_3) = 0 = \phi x \qquad (1)$$

where x_1, x_2 and x_3 are running coordinates. We now wish to express the condition that this first polar (1) shall have a double point, namely that the first partial derivatives shall vanish. We have then

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_1} &= y_1 \frac{\partial^2 f}{\partial x_1^2} + y_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_3 \frac{\partial^2 f}{\partial x_1 \partial x_3} = 0 \\
\frac{\partial \Phi}{\partial x_2} &= y_1 \frac{\partial^2 f}{\partial x_2 \partial x_1} + y_2 \frac{\partial^2 f}{\partial x_2^2} + y_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} = 0 \qquad (2) \\
\frac{\partial \Phi}{\partial x_3} &= y_1 \frac{\partial^2 f}{\partial x_3 \partial x_1} + y_2 \frac{\partial^2 f}{\partial x_3 \partial x_2} + y_3 \frac{\partial^2 f}{\partial x_3^2} = 0
\end{aligned}$$

Eliminating y_1, y_2 and y_3 from equations (2) we have the locus of the double points of first polars, viz.–

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix} = 0$$

which is the equation of the Hessian.

Theorem 8. — For the cubic, the Hessian and the Steinerian are identical.

Proof:-

We have defined the Hessian as the locus of all points whose polar conics break up into pairs of right lines, and the Steinerian as the locus of all points whose first polars have double points. But, for the cubic the first polar is of degree (3-1), or is the polar conic. Hence the Steinerian of the cubic is the locus of all points whose polar conics have double points or, in other words, break up into

pairs of right lines. Therefore the Hessian and the Steinerian of the cubic are the same.

Example 8.

Let us refer to the curve

$$x^3 + y^3 - 3xyz = 0$$

of Example 2. page 7. and Example 5. page 29. In Example 2. the Steinerian of this curve was found to be

$$x^3 + y^3 + xyz = 0$$

and in Example 5, the Hessian was found to be the same.

Theorem 10.– The general curve of the third order is the Hessian of three other cubics belonging to the same syzygetic system.

Proof:–

Any cubic of a syzygetic system is a cubic thro' the nine inflections of a

given cubic, or thro' the intersections of the given cubic with its Hessian.

Now let the equation of the cubic be

$$f = 0 \tag{1}$$

and that of its Hessian be

$$H = 0 \tag{2}$$

Then any curve thro' the intersections of these two may be represented by the equation

$$\kappa f + \lambda H = 0 \tag{3}$$

Now forming the Hessian of this curve we have another equation of the same type as (3), viz.-

$$Kf + LH = 0 \tag{4}$$

in which κ and λ enter in the third degree. Then we may write

$$K = \phi^{(3)}(\kappa, \lambda)$$

and $\qquad\qquad\qquad\qquad$ (5)

$$L = \varphi^{(3)}(\kappa, \lambda)$$

hence

$$\frac{K}{L} = \varphi^{(3)}\left(\frac{\kappa}{\lambda}\right) \qquad (6)$$

Thus we have three values of $\frac{\kappa}{\lambda}$ and hence will get three curves of the type $\kappa f + \lambda H = 0$ for which $Kf + LH = 0$ is the Hessian.

The Canonical Form of the Equation of the Hessian.

It is shown by Clebsch: "Vorlesungen über Geometrie", that if the inflections triangle is taken as the coordinate triangle, the equations of all cubics can be reduced to the following, canonical, form.

$$a(x^3+y^3+z^3) + 6bxyz = 0 \qquad (1)$$

Let us find the Hessian of this cubic. Proceeding in the usual manner, we have

$$\begin{vmatrix} ax & bg & by \\ bz & ay & bx \\ by & bx & az \end{vmatrix} = \frac{1}{6} H \qquad (2)$$

The expansion of this determinant is

$$a^3 + 2b^3xyz - ab^2(x^3+y^3+z^3) = 0 \qquad (3)$$

or

$$\lambda(x^3+y^3+z^3) + 4\beta xyz = 0 \qquad (4)$$

the canonical form of the Hessian.

Singularities of the Hessian of a Non-singular n-ic.

In the following we shall denote the order of the Hessian by n', its class by κ', number of nodes by d', of cusps by c', of double tangents by t' and of inflections by i'. We shall deal with a non-singular ground curve of order n, so that $d=0$ and $c=0$

It has already been shown, page 19,

that the order of the Hessian is $n' = 3(n-2)$.
Since the ground curve is non-singular, its Hessian is also non-singular, so that we also have $d' = 0$ and $c' = 0$. Now Plücker's equations are:–

$$n(n-1) = k + 2d + 3c$$

$$k(k-1) = n + 2t + 3i$$

$$3n(n-2) = i + 6d + 8c$$

$$3k(k-2) = c + 6t + 8i$$

From these we have:–

$$i' = 3(n'(n'-2) - 6d' - 8c' = 3n'(n'-2) = 9(n-2)(3n-8)$$

$$k' = n'(n'-1) - 2d' - 3c' = n'(n'-1) = 3(n-2)(3n-7)$$

$$t' = 3k'(k'-2) - c' - 8i' = 3k'(k'-2) = \frac{27}{2}(n-1)(n-2)(n-3)(3n-8).$$

The deficiency of the Hessian is given by

$$p' = \tfrac{1}{2}(n'-1)(n'-2) = \tfrac{1}{2}(3n-7)(3n-8)$$

Let us now obtain the <u>Order</u> and <u>Class</u> of the <u>Steinerian</u> of a <u>non-singular ground curve.</u> – (Clebsch: [illegible] lesungen

über Geometrie.)

As has been shown, the equation of the Hessian is obtained by eliminating y_1, y_2 and y_3 from the equations –

$$a_x^{n-2} a_y a_1 \equiv y_1 \frac{\partial^2 f}{\partial x_1^2} + y_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_3 \frac{\partial^2 f}{\partial x_1 \partial x_3} = 0$$

$$a_x^{n-2} a_y a_2 \equiv y_1 \frac{\partial^2 f}{\partial x_2 \partial x_1} + y_2 \frac{\partial^2 f}{\partial x_2^2} + y_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} = 0 \qquad (1)$$

$$a_x^{n-2} a_y a_3 \equiv y_1 \frac{\partial^2 f}{\partial x_3 \partial x_1} + y_2 \frac{\partial^2 f}{\partial x_3 \partial x_2} + y_3 \frac{\partial^2 f}{\partial x_3^2} = 0$$

The equation of the Steinerian on the other hand is obtained by eliminating x_1, x_2 and x_3 from the same equations. Each of the equations is of degree $(n-2)$ in the x's, and the degree of the resulting equation is therefore $3(n-2)^2$. [see Clebsch, p 313].

To find the class of the Steinerian is a more difficult matter. In order to do so, we must find the total number of tangents that can be drawn from

any point to the curve. Now the coordinates of any tangent can be found from the equations -

$$u_1 y_1 + u_2 y_2 + u_3 y_3 = 0$$
$$u_1 \partial y_1 + u_2 \partial y_2 + u_3 \partial y_3 = 0 \qquad (2)$$

of two consecutive points of the curve, since the tangent may be considered as the line joining two consecutive points of the curve. Now the points (y_1, y_2, y_3) and $(y_1 + \partial y_1, y_2 + \partial y_2, y_3 + dy_3)$ not only satisfy equations (2) but must also satisfy equations (1), provided that in these we replace x_1 by $x_1 + \partial x_1$, x_2 by $x_2 + \partial x_2$ and x_3 by $x_3 + \partial x_3$. For convenience we denote with Clebsch $\frac{\partial^2 f}{\partial x_1 \partial x_k}$ by f_{1k}. Making the above substitutions in (1) we then have

$$(y_1 + \partial y_1)(f_{11} + \partial f_{11}) + (y_2 + \partial y_2)(f_{12} + \partial f_{12}) + (y_3 + \partial y_3)(f_{13} + \partial f_{13}) =$$

$$y_1 f_{11} + y_1 \partial f_{11} + f_{11} \partial y_1 + \partial y_1 \partial f_{11} + y_2 f_{12} + y_2 \partial f_{12} + f_{12} \partial y_2 +$$

$$\partial y_2 \partial f_{12} + y_3 f_{13} + y_3 \partial f_{13} + f_{13} \partial y_3 + \partial y_3 \partial f_{13}.$$

from the first equation. Remembering that infinitesmals of higher order drop out, the complete substitution gives

$$f_{11}\,\partial y_1 + f_{12}\,\partial y_2 + f_{13}\,\partial y_3 + y_1\,\partial f_{11} + y_2\,\partial f_{12} + y_3\,\partial f_{13} = 0$$
$$f_{21}\,\partial y_1 + f_{22}\,\partial y_2 + f_{23}\,\partial y_3 + y_1\,\partial f_{21} + y_2\,\partial f_{22} + y_3\,\partial f_{23} = 0 \qquad (3)$$
$$f_{31}\,\partial y_1 + f_{32}\,\partial y_2 + f_{33}\,\partial y_3 + y_1\,\partial f_{31} + y_2\,\partial f_{32} + y_3\,\partial f_{33} = 0$$

Let us now multiply the first of the equations under (1) by x_1, the second by x_2, and the third by x_3 and add the results. This gives us –

$$y_1\left[x_1\frac{\partial^2 f}{\partial x_1^2} + x_2\frac{\partial^2 f}{\partial x_2\partial x_1} + x_3\frac{\partial^2 f}{\partial x_3\partial x_1}\right] + y_2\left[x_1\frac{\partial^2 f}{\partial x_1\partial x_2} + x_2\frac{\partial^2 f}{\partial x_2^2} + x_3\frac{\partial^2 f}{\partial x_3\partial x_2}\right] +$$
$$y_3\left[x_1\frac{\partial^2 f}{\partial x_1\partial x_3} + x_2\frac{\partial^2 f}{\partial x_2\partial x_3} + x_3\frac{\partial^2 f}{\partial x_3^2}\right] = 0 \qquad (4)$$

now

$$x_1\frac{\partial^2 f}{\partial x_1^2} + x_2\frac{\partial^2 f}{\partial x_2\partial x_1} + x_3\frac{\partial^2 f}{\partial x_3\partial x_1} = \frac{\partial}{\partial x_1}\left[x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3}\right] - \frac{\partial f}{\partial x_1}$$
$$= \frac{\partial}{\partial x_1}\left[nf\right] - \frac{\partial f}{\partial x_1}$$
$$= (n-1)\frac{\partial f}{\partial x_1} = (n-1)f_1$$

For the other square brackets we obtain

similar results, so that (4) goes over into

$$(n-1)y_1f_1 + (n-1)y_2f_2 + (n-1)y_3f_3 = 0$$

or

$$y_1f_1 + y_2f_2 + y_3f_3 = 0 \tag{5}$$

If we multiply the corresponding equations of (1) by ∂x_1, ∂x_2 and ∂x_3 respectively, we obtain in a similar manner –

$$y_1\partial f_1 + y_2\partial f_2 + y_3\partial f_3 = 0 \tag{6}$$

Again multiply the first of the equations under (3) by x_1, the second by x_2 and the third by x_3 and add the results. We then have

$$\partial y_1\left[x_1f_{11} + x_2f_{21} + x_3f_{31}\right) + \partial y_2\left[x_1f_{21} + x_2f_{22} + x_3f_{33}\right] + \partial y_3\left[x_1f_{31} + x_2f_{32} + x_3f_{33}\right] + y_1\left[x_1\partial f_{11} + x_2\partial f_{21} + x_3\partial f_{31}\right] + y_2\left[x_1\partial f_{12} + x_2\partial f_{22} + x_3\partial f_{32}\right] + y_3\left[x_1\partial f_{13} + x_2\partial f_{23} + x_3\partial f_{33}\right] = 0$$

Hence

$$f_1\partial y_1 + f_2\partial y_2 + f_3\partial y_3 = 0 \tag{7}$$

Comparing this equation with (2) we have

$$\mu u_1 = f_1, \quad \mu u_2 = f_2, \quad \mu u_3 = f_3 \tag{8}$$

This enables us to express analytically, the coordinates of a tangent to the Steinerian in terms of the corresponding point of the Hessian. Eliminating the x's from these equations taken in conjunction with the equation of the Hessian, gives us the equation of the Steinerian in line coordinates. This equation is evidently of degree $3(n-2)(n-1)$, since the Hessian is of degree $3(n-2)$ in the x's and (8) of degree $(n-1)$. Hence $3(n-2)(n-1)$ is the class of the Steinerian.

Applying this to a curve of degree three we see that the Steinerian is of class three.

Theorem 11. – The deficiency of the Hessian, Steinerian and Cayleyan is the same.

Proof: –

We have already shown that there is a one to one correspondence between the points of these three curves. that is to say

to each point of the one curve there corresponds a point of the other. Now Clebsch (page 458) proves that any two algebraic curves which stand in this one to one correspondence, have the same deficiency. Hence the Hessian, Steinerian and Cayleyan have the same deficiency.

Singularities of the Steinerian of a non-singular Curve.

We have already, page 46, obtained from Plückers formulae, the number of singularities of a non-singular curve. Let us find these for the Steinerian. The deficiency of the Hessian is $\frac{1}{2}(3n-7)(3n-8)$ and hence this is also the deficiency of the Steinerian. For the Steinerian we have

$$p' = \tfrac{1}{2}(3n-7)(3n-8)$$

$$n' = 3(n-2)^2$$

$$k' = 3(n-1)(n-2)$$

where n is the order of the ground curve. If now we denote the number of nodes of the Steinerian by d' and the number of cusps by c', we have the following equations for determining c' and d'.

$$p' = \tfrac{1}{2}(n'-1)(n'-2) - (d'+c')$$

$$n'(n'-1) = K' + 2d' + 3c'$$

Solving these for c' and d' and replacing p', n' and K' by their values found above, we have

$$d' = \tfrac{3}{2}(n-2)(n-3)(3n^2-9n-5)$$

$$c' = 12(n-2)(n-3)$$

We have also from Plückers equations

$$K'(K'-1) = n' + 2t' + 3i'$$

$$3K'(K'-2) = c' + 6t' + 8i'$$

where i' is the number of inflections and t' the number of double tangents to the Steinerian. Solving these we have

$$t' = \tfrac{3}{2}(n-2)(n-3)(3n^2-3n-8)$$

$$i = 3(n-2)(4n-9)$$

We see from the above that the Steinerian may have nodes and cusps. A node on the Steinerian may arise if to two distinct points of the Hessian, there corresponds the same point of the Steinerian. Conversely, if the Steinerian has a double point, then to this corresponds two distinct points of the Hessian.

Theorem 12. – The Steinerian is the envelope of the linear polars of points of the Hessian. These linear polars are tangent to the Steinerian at the points corresponding to the poles on the Hessian.

Proof:–

The linear polar of a point on the Hessian is

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3} \right) f = 0 \qquad (1)$$

Substituting the values of the partial derivatives found in equation (8), page 51,

equation (1) becomes

$$u_1 y_1 + u_2 y_2 + u_3 y_3 = 0 \qquad (2)$$

which is, as we have seen on page 49, the equation of the tangent to the Steinerian. Therefore the Steinerian is tangent to the linear polars of the Hessian.

Theorem 13. - The Steinerian is tangent to all inflectionall tangents of the ground curve.

Proof: -

The Steinerian is tangent to the linear of a point of inflection of the ground curve, for we know from the theorem 12 just proved that the Steinerian is tangent to all linear polars of points of the Hessian, and a point of inflection is a point of the Hessian. But we know that the linear polar of a point of inflection is the inflectional

tangent itself. Therefore the Steinerian is tangent to the inflectional tangent.

In the preceding are given some of the most important properties of the Hessian and Steinerian. The following bibliography is as complete as possible down to date.

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